REAL ANALYSIS PRELIMINARY EXAM
AUGUST 22, 2026

All responses require justification.

(1) Suppose (x_n) is a sequence in $\mathbb{R}^m$ for some natural number m, with $|x_{n+1} - x_n| \leq \dfrac{1}{n^2+n}$ for all n. Prove that this sequence converges.

(2) Prove or disprove the claim that the function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$, if

$$f(x,y) = \begin{cases} e^{(x^2y+xy^2)/(x^2+y^2)}, & (x,y) \neq (0,0), \\ 1, & (x,y) = (0,0). \end{cases}$$

(3) Suppose that $f(x) : [a,b] \to \mathbb{R}$ is a function that is non-negative for all $x \in [a,b]$, Riemann integrable on $[a,b]$, and there exists a point $c \in [a,b]$ such that $f(c) > 0$ and $f(x)$ is continuous at c. Prove that $\displaystyle\int_a^b f(x)dx > 0$.

(4) Let $\omega = \dfrac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.

(a) Use the divergence theorem to find the integral of ω over a sphere of radius 1 centered at $(0,0,2)$.

(b) Find the integral of ω over a sphere of radius 1 centered at the origin.

(5) (a) Find all real values α so that $\displaystyle\sum_{n=1}^{\infty} \frac{n^\alpha}{1+(x-n)^2}$ defines a function $f(x) : \mathbb{R} \to \mathbb{R}$.

(b) Show that if $f(x) : \mathbb{R} \to \mathbb{R}$ is defined, it is also continuous.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be the function of period 2π such that $f(x) = 1$ for $-\pi < x < 0$ and $f(x) = 2$ for $0 \leq x < \pi$.

(a) Find the Fourier series of f.

(b) For which $x \in \mathbb{R}$ does the Fourier series converge to $f(x)$? Justify your answer, and state the values to which the series converges at the remaining points.

(c) Use Parseval's identity to evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

(7) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable with $f(0) = 0$ and $f'(0) \neq 0$. Consider the equation

$$f(x) = tg(x),$$

where $t \in \mathbb{R}$. Prove that there exists a $\delta > 0$ such that in $(-\delta, \delta)$, there is a unique continuous function $x = x(t)$ satisfying the equation above and $x(0) = 0$.

(8) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and both limits $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on $\mathbb{R}$.